FORMALITY AND HARD LEFSCHETZ PROPERTIES OF ASPHERICAL MANIFOLDS

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ABSTRACT. For a virtually polycyclic group Γ, we consider an aspherical manifold M_Γ with $\pi_1(M_\Gamma) = \Gamma$ constructed by Baues. For a Lie group $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ with the action $\phi : \mathbb{R}^n \to \operatorname{Aut} \mathbb{R}^m$ is semi-simple, we show that if Γ is a finite extension of a lattice of G then M_Γ is formal. Moreover if M_Γ admits a symplectic structure, we show M_Γ satisfies the hard Lefschetz property. By those results we give many examples of formal solvmanifolds satisfying the hard Lefschetz properties but not admitting Kähler structures.

1. Introduction

Formal spaces in the sense of Sullivan are important for de Rham homotopy theory. Famous examples of formal spaces are compact Kähler manifolds (see [7]). Suppose Γ is a torsion-free finitely generated nilpotent group. Then $K(\Gamma, 1)$ is formal if and only if Γ is abelian by Hasegawa's theorem in [9]. But in case Γ is a virtually polycyclic group, the formality of $K(\Gamma, 1)$ is more complicated. One of the purposes of this paper is to apply the way of the algebraic hull of Γ to study the formality of $K(\Gamma, 1)$. For a torsion-free virtually polycyclic group Γ, we have a unique algebraic group $\mathbf{H}_\Gamma$ with an injective homomorphism $\psi : \Gamma \to \mathbf{H}_\Gamma$ so that:

(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_\Gamma$.
(2) The centralizer $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma))$ of $\mathbf{U}(\mathbf{H}_\Gamma)$ is contained in $\mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma) = \operatorname{rank} \Gamma$.

Such $\mathbf{H}_\Gamma$ is called the algebraic hull of Γ. We call the unipotent radical of $\mathbf{H}_\Gamma$ the unipotent hull of Γ and denote it by $\mathbf{U}_\Gamma$. In [3], Baues constructed a compact aspherical manifold M_Γ with the fundamental group Γ called the standard Γ-manifold by the algebraic hull of Γ. And he gave the way of computation of the de Rham cohomology of M_Γ. By the application of these results, we prove:

2010 *Mathematics Subject Classification.* 20F16; 22E40; 32J27; 55P20; 55P62.
Key words and phrases. polycyclic group, algebraic hull, formality, solvmanifold, the hard Lefschetz property.

Proposition 1.1. *If the unipotent hull* $\mathbf{U}_{\boldsymbol{\Gamma}}$ *of* Γ *is abelian,* $K(\Gamma, 1)$ *is formal.*

So we are interested in knowing criteria for $\mathbf{U}_\Gamma$ to be abelian. We prove the following theorem.

Theorem 1.2. *Let* Γ *be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent:*
(1) $\mathbf{U}_\Gamma$ *is abelian.*
(2) Γ *is a finite extension group of a lattice of a Lie group* $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*

Therefore we have:

Corollary 1.3. *If* Γ *satisfies the condition* (2) *in Theorem 1.2, then* $K(\Gamma, 1)$ *is formal.*

By this corollary, we have examples of formal spaces which give a relation to the geometries of 3-dimensional manifolds.

Corollary 1.4. *Let* M *be a compact* 3*-dimensional manifold. If the geometric structure of* M *is* E^3 *or* Sol*, then* M *is formal.*

As in the case of formality the hard Lefschetz properties are important properties of compact Kähler manifolds. We have the following proposition.

Proposition 1.5. *Suppose the standard* Γ*-manifold* M_Γ *admits a symplectic structure. If the unipotent hull* $\mathbf{U}_\Gamma$ *is abelian,* M_Γ *satisfies the hard Lefschetz property.*

A solvmanifold is an example of standard Γ-manifold M_Γ. For a nilmanifold M, if M is formal or satisfies the hard Lefschetz property, then M is diffeomorphic to a torus(see [4] [9]). But by the result of this paper and Arapura's theorem in [1], we have many examples of formal solvmanifolds satisfying the hard Lefschetz properties but not admitting Kähler structures.

Corollary 1.6. *Let* $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple and* G *is not type (I) i.e. for any* $g \in G$ *the all eigenvalues of* Ad_g *have absolute value 1. Then for any lattice* Γ *of* G*,* G/Γ *is formal solvmanifolds which does not admit a Kähler structure. If* G/Γ *admits a symplectic structure,* G/Γ *satisfies the hard Lefschetz property.*

The paper is organized in the following way. In Section 2 the Preliminaries for this paper are written. In Section 3 we review the algebraic hulls of virtually polycyclic groups or solvable Lie groups. We compile some facts in [3] and [15]. In Section 4 we prove Theorem 1.2. The

idea of this section is to apply the embeddings solvable Lie algebras in splittable Lie algebras. In Section 5 we prove Proposition 1.1 and Corollary 1.4. In Section 6 we prove Proposition 1.5 and Corollary 1.6. In Section 7 we give an example of a formal standard Γ-manifold with the hard Lefschetz property such that U_Γ is not abelian.

2. Preliminaries

2.1. **Algebraic groups.** Let k be a subfield of $\mathbb{C}$. A group $\mathbf{G}$ is called k-algebraic group if $\mathbf{G}$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in k. Let $\mathbf{G}(k)$ denote the set of k-points of $\mathbf{G}$ and $\mathbf{U}(\mathbf{G})$ the maximal Zariski-closed unipotent normal k-subgroup of $\mathbf{G}$ called the unipotent radical of $\mathbf{G}$. In this paper, algebraic groups are always written in the bold face.

2.2. **Nilpotent Lie algebras and $\mathbb{R}$-unipotent algebraic groups.** Let N be a simply connected Lie group and $\mathfrak{n}$ the Lie algebra of N, By the Baker-Campbell-Hausdorff formula, the exponential map $\exp : \mathfrak{n} \to N$ is a diffeomorphism and we have the group structure on $\mathfrak{n}$ induced by N such that

$$X \cdot Y = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X-Y,[X,Y]] + \cdots .$$

Since $\mathfrak{n}$ is nilpotent, $X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X-Y,[X,Y]]\cdots$ is a finite sum and given by polynomial functions on $\mathfrak{n} \cong \mathbb{R}^{\dim \mathfrak{n}}$. So we have an $\mathbb{R}$- algebraic group structure on $\mathfrak{n}_{\mathbb{C}} \cong \mathbb{C}^{\dim \mathfrak{n}}$ such that $\mathfrak{n}_{\mathbb{C}}(\mathbb{R}) = \mathfrak{n}$.

Let $\mathbf{U_n}(\mathbb{C})$ be the upper triangular matrices and $\mathfrak{u}_n(\mathbb{C})$ Lie algebra of $\mathbf{U_n}(\mathbb{C})$. Then the exponential map $\exp : \mathfrak{u}_n(\mathbb{C}) \to \mathbf{U_n}(\mathbb{C})$ gives the isomorphism of $\mathbb{R}$-algebraic groups $\mathfrak{u}_n(\mathbb{C})$ and $\mathbf{U_n}(\mathbb{C})$ For any Lie subalgebra $\mathfrak{u}$ of $\mathfrak{u}_n(\mathbb{R})$, we have the faithful representation of an $\mathbb{R}$-algebraic group $\exp : \mathfrak{u} \otimes \mathbb{C} \to \mathbf{U_n}(\mathbb{C})$. So we have the 1-1 correspondence between Lie sub-algebras of $\mathfrak{u}_n(\mathbb{R})$ and $\mathbb{R}$-algebraic subgroups of $\mathbf{U_n}(\mathbb{C})$. By Engel's theorem, any nilpotent Lie algebra is a Lie subalgebra of $\mathfrak{u}_n(\mathbb{R})$ for some n. Hence we have:

Proposition 2.1. *By the exponential map, we have the 1-1 correspondence between nilpotent Lie algebras and unipotent $\mathbb{R}$-algebraic groups.*

Let $\mathfrak{n}_1$, $\mathfrak{n}_2$ be nilpotent Lie algebras . Let $\mathbf{N_1}$, $\mathbf{N_2}$ be the $\mathbb{R}$-unipotent groups which correspond to $\mathfrak{n}_1$, $\mathfrak{n}_2$. Let $f : \mathfrak{n}_1 \to \mathfrak{n}_2$ be a Lie algebra homomorphism. Since f is a linear map, $f : \mathfrak{n}_1 \otimes \mathbb{C} \to \mathfrak{n}_2 \otimes \mathbb{C}$ is a $\mathbb{R}$-algebraic group homomorphism. By $f \to \exp \circ f \circ \exp^{-1}$, we have the 1-1 correspondence between Lie algebra homomorphisms $\mathfrak{n}_1 \to \mathfrak{n}_2$ and $\mathbb{R}$-algebraic group homomorphisms $\mathbf{N_1} \to \mathbf{N_2}$.

For a nilpotent Lie algebra $\mathfrak{n}$, the group of automorphisms $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$ is an $\mathbb{R}$-algebraic group with $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})(\mathbb{R}) = \mathrm{Aut}(\mathfrak{n})$ by the Lie bracket on $\mathfrak{n}$. Let $\mathbf{N}$ be the $\mathbb{R}$-algebraic group which corresponds to $\mathfrak{n}$. Let $\mathrm{Aut}_{\mathrm{a}}(\mathbf{N})$ denote the group of automorphisms of $\mathbf{N}$. By the above correspondence, we identify $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$ with $\mathrm{Aut}_a(\mathbf{N})$.

3. **Algebraic hulls**

In this section we explain the algebraic hulls of polycyclic groups or simply connected solvable Lie groups. We compile some facts in [3] and [15] and prove some lemmas to prove the main theorem.

3.1. **Polycyclic groups and simply connected solvable Lie groups.**

We first review basic informations of polycyclic groups. See [15] and [18] for more details.

Definition 3.1. A group Γ is polycyclic if it admits a sequence

$$\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_k = \{e\}$$

of subgroups such that each Γ_i is normal in Γ_{i-1} and Γ_{i-1}/Γ_i is cyclic. We denote $\mathrm{rank}\,\Gamma = \sum_{i=1}^{i=k} \mathrm{rank}\,\Gamma_{i-1}/\Gamma_i$.

We have relations between polycyclic groups and solvable Lie groups. We have the following two theorem.

Theorem 3.2. ([15, Proposition 3.7, Theorem 4.28]) *Let G be a simply connected solvable Lie group and Γ be a lattice in G. Then Γ is torsion-free polycyclic and* $\dim G = \mathrm{rank}\,\Gamma$*. Conversely every polycyclic group admits a finite index normal subgroup which is isomorphic to a lattice in a simply connected solvable Lie group.*

Let Γ be a virtually polycyclic group and Γ' be a finite index polycyclic subgroup. We denote $\mathrm{rank}\,\Gamma = \mathrm{rank}\,\Gamma'$.

3.2. **Algebraic hulls of torsion-free virtually polycyclic groups.**

Let k be a subfield of $\mathbb{C}$. Let Γ be a torsion-free virtually polycyclic group.

Definition 3.3. We call a k-algebraic group $\mathbf{H}_\Gamma$ a k-algebraic hull of Γ if there exists an injective group homomorphism $\psi : \Gamma \to \mathbf{H}_\Gamma(k)$ and $\mathbf{H}_\Gamma$ satisfies the following conditions:
(1) $\psi(\Gamma)$ is Zariski-dense in $\mathbf{H}_{\boldsymbol{\Gamma}}$.
(2) $Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma)$.
(3) $\dim \mathbf{U}(\mathbf{H}_\Gamma)$=$\mathrm{rank}\,\Gamma$.

Theorem 3.4. ([3, Theorem A.1]) *There exists a k-algebraic hull of Γ and a k-algebraic hull of Γ is unique up to k-algebraic group isomorphism.*

Let $\mathbf{H}_\Gamma$ be the k-algebraic hull of Γ and let $\mathbf{U}_\Gamma$ be the unipotent radical of $\mathbf{H}_\Gamma$. We call $\mathbf{U}_{\mathbf{\Gamma}}$ the k-unipotent hull of Γ.

Let Γ be a torsion-free virtually polycyclic group and $\Delta \subset \Gamma$ be a finite index subgroup of Γ. Let $\mathbf{H}_\Gamma$ be the k-algebraic hull of Γ and $\mathbf{G}$ the Zariski-closure of $\psi(\Delta)$ in $\mathbf{H}_\Gamma$.

Lemma 3.5. *The algebraic group $\mathbf{G}$ is the k-algebraic hull of Δ and we have $\mathbf{U}_\Delta = \mathbf{U}_\Gamma$.*

Proof. Let $\mathbf{H}^0_\Gamma$ be the identity component of $\mathbf{H}_\Gamma$. Since $\mathbf{G}$ is a closed finite index subgroup of $\mathbf{H}_\Gamma$, we have $\mathbf{H}^0_\Gamma \subset \mathbf{G}$. Since Γ is virtually polycyclic, $\mathbf{H}^0_\Gamma$ is solvable. Hence we have $\mathbf{U}(\mathbf{H}_\Gamma) = (\mathbf{H}^0_\Gamma)_{unip} = \mathbf{U}(\mathbf{G})$. Since $\operatorname{rank}\Gamma = \operatorname{rank}\Delta$, we have

$$\dim \mathbf{U}(\mathbf{G}) = \operatorname{rank}\Delta.$$

And we have

$$Z_{\mathbf{G}'}(\mathbf{U}(\mathbf{G})) \subset Z_{\mathbf{H}_\Gamma}(\mathbf{U}(\mathbf{H}_\Gamma)) \subset \mathbf{U}(\mathbf{H}_\Gamma) = \mathbf{U}(\mathbf{G}).$$

Hence the lemma follows. □

3.3. **Algebraic hulls of simply connected solvable Lie groups.** Let G be a simply connected solvable $\mathbb{R}$-Lie group. Let k be a subfield of $\mathbb{C}$ which contains $\mathbb{R}$ as a subfield.

Lemma 3.6. ([15, Lemma 4.36]) *Let $\rho : G \to GL_n(\mathbb{C})$ be a representation and $\mathbf{G}$ be the Zariski-closure of $\rho(G)$ in $GL_n(\mathbb{C})$. Then we have*

$$\dim \mathbf{U}(\mathbf{G}) \le \dim G.$$

Definition 3.7. We call a k-algebraic group $\mathbf{H}_G$ a k-algebraic hull of G if there exists an injective Lie group homomorphism $\psi : G \to \mathbf{H}_\Gamma(k)$ and satisfies the following conditions:
(1) $\psi(G)$ is Zariski-dense in $\mathbf{H}_G$.
(2) $Z_{\mathbf{H}_G}(\mathbf{U}(\mathbf{H}_G)) \subset \mathbf{U}(\mathbf{H}_G)$.
(3) $\dim \mathbf{U}(\mathbf{H}_G) = \dim G$.

Theorem 3.8. ([15, Proposition 4.4]) *There exists a k-algebraic hull of G and a $\mathbb{R}$-algebraic hull of G is unique up to k-algebraic group isomorphism.*

We call the unipotent radical of the k-algebraic hull of G the k-unipotent hull of G and denote $\mathbf{U}_G$.

Suppose G has a lattice Γ.

Theorem 3.9. ([15, Theorem 3.2]) *Let $\rho : G \to GL_n(\mathbb{C})$ be a representation and let $\mathbf{G}$ and $\mathbf{G}'$ be the Zariski-closures of $\rho(G)$ and $\rho(\Gamma)$ in $GL_n(\mathbb{C})$. Then we have $\mathbf{U}(\mathbf{G}) = \mathbf{U}(\mathbf{G}')$.*

Let $\psi : G \to \mathbf{H}_G$ be the $\mathbb{R}$-algebraic hull of G and $\mathbf{H}'$ be the Zariski-closure of $\psi(\Gamma)$ in $\mathbf{H_G}$.

Lemma 3.10. *$\mathbf{H}'$ is the $\mathbb{R}$-algebraic hull of Γ and we have $\mathbf{U}_G = \mathbf{U}_\Gamma$.*

Proof. By Theorem 3.9, $\mathbf{U}(\mathbf{H}_G) = \mathbf{U}(\mathbf{H}')$. So we have

$$\dim \mathbf{U}(\mathbf{H}') = \dim \mathbf{U}(\mathbf{H}_G) = \dim G = \operatorname{rank} \Gamma$$

and

$$Z_{\mathbf{H}'}(\mathbf{U}(\mathbf{H}')) \subset Z_{\mathbf{H}_G}(\mathbf{U}(\mathbf{H}_G)) \subset \mathbf{U}(\mathbf{H}_G) \subset \mathbf{U}(\mathbf{H}').$$

Hence the lemma follows.

□

3.4. **Cohomology computations of aspherical manifolds with virtually torsion-free polycyclic fundamental groups.** Let Γ be a torsion-free virtually polycyclic group. and $\mathbf{H_\Gamma}$ be the $\mathbb{Q}$-algebraic hull of Γ. Denote $H_\Gamma = \mathbf{H_\Gamma}(\mathbb{R})$. Let U_Γ be the unipotent radical of H_Γ and let T be a maximal reductive subgroup. Then H_Γ decomposes as a semi-direct product $H_\Gamma = T \ltimes U_\Gamma$. Let $\mathfrak{u}$ be the Lie algebra of U_Γ. Since the exponential map $\exp : \mathfrak{u} \longrightarrow U_\Gamma$ is a diffeomorphism, U_Γ is diffeomorphic to $\mathbb{R}^n$ such that $n = \operatorname{rank} \Gamma$. The splitting $H_\Gamma = T \ltimes U_\Gamma$ gives rise to the affine action $\alpha : H_\Gamma \longrightarrow \operatorname{Aut}(U_\Gamma) \ltimes U_\Gamma$ such that α is an injective homomorphism.

In [3] Baues constructed a compact aspherical manifold $M_\Gamma = \alpha(\Gamma)\backslash U_\Gamma$ with $\pi_1(M_\Gamma) = \Gamma$. We call M_Γ a standard Γ-manifold.

Theorem 3.11. ([3, Theorem 1.2]) *Standard Γ-manifold is unique up to diffeomorphism.*

Let $A^*(M_\Gamma)$ be the de Rham complex of M_Γ. Then $A^*(M_\Gamma)$ is the set of the Γ-invariant differential forms $A^*(U_\Gamma)^\Gamma$ on U_Γ. Let $(\bigwedge \mathfrak{u}^*)^T$ be the left-invariant forms on U_Γ which are fixed by T. Since $\Gamma \subset H_\Gamma = T \ltimes U_\Gamma$, we have the inclusion

$$(\bigwedge \mathfrak{u}^*)^T = A^*(U_\Gamma)^{H_\Gamma} \subset A^*(U_\Gamma)^\Gamma = A^*(M_\Gamma).$$

Theorem 3.12. ([3, Theorem 1.8]) *This inclusion induces a cohomology isomorphism.*

4. Constructions of algebraic hulls

4.1. The embedings of solvable Lie algebras in splittable Lie algebras. The idea of this subsection is based on [16]. Let $\mathfrak{g}$ be a solvable Lie algebra, and $\mathfrak{n} = \{X \in \mathfrak{g} | \mathrm{ad}_X \text{ is nilpotent}\}$. $\mathfrak{n}$ is the maximal nilpotent ideal of $\mathfrak{g}$ and called the nilradical of $\mathfrak{g}$.

Lemma 4.1. ([12, p.58]) *We have* $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{n}$.

Let $D(\mathfrak{g})$ be the derivation of $\mathfrak{g}$. By the Jordan decomposition, we consider $\mathrm{ad}_X = d_X + n_X$ such that d_X is a semi-simple operator and n_X is a nilpotent operator.

Lemma 4.2. ([16, Proposition 3]) *We have* $d_X,\ n_X \in D(\mathfrak{g})$.

Then we have the homomorphism $f : \mathfrak{g} \to D(\mathfrak{g})$ such that $f(X) = d_X$ for $X \in \mathfrak{g}$. Since $\mathrm{ker} f = \mathfrak{n}$, we have $\mathrm{Im} f \cong \mathfrak{g}/\mathfrak{n}$.

Let $\bar{\mathfrak{g}} = \mathrm{Im} f \ltimes \mathfrak{g}$. Let $\bar{\mathfrak{n}} = \{X - d_X \in \bar{\mathfrak{g}} | X \in \mathfrak{g}\}$. Since $\mathrm{ad}_{X-d_X} = \mathrm{ad}_X - d_X$ on $\mathfrak{g}$, ad_{X-d_X} is a nilpotent operator. So $\bar{\mathfrak{n}}$ consists of nilpotent elements.

Proposition 4.3. *We have* $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$ *for any* $X \in \mathfrak{g}$, $\bar{\mathfrak{n}}$ *is a nilpotent ideal of* $\bar{\mathfrak{g}}$ *and* $\bar{\mathfrak{g}} = \mathrm{Im} f \ltimes \bar{\mathfrak{n}}$.

Proof. By Lie's theorem, we choose a basis $X_1, \dots, X_l$ of $\mathfrak{n} \otimes \mathbb{C}$ such that $\mathrm{ad}_{\mathfrak{g}}$ on $\mathfrak{n}$ are represented by upper triangular matrices. For any $X \in \mathfrak{g}$, we have

$\mathrm{ad}_X(X_1) = a_{X,1} X_1$
$\mathrm{ad}_X(X_2) = a_{X,2} + b_{X,12} X_1$
$\vdots$
$\mathrm{ad}_X(X_l) = a_{X,l} X_l + b_{X,l-1l} X_{l-1} + \cdots + b_{X,1l} X_1$.

We take a basis $X_1, \dots X_l, X_{l+1}, \dots X_{l+m}$ of $\mathfrak{g} \otimes \mathbb{C}$. By Lemma 4.1, $\mathrm{ad}_X(X_i) \in \mathfrak{n}$. Hence we have

$\mathrm{ad}_X(X_{l+1}) = b_{X,ll+1} X_l + \cdots + b_{X,1l+1} X_1$
$\vdots$
$\mathrm{ad}_X(X_{l+m}) = b_{X,ll+m} X_l + \cdots + b_{X,1l+m} X_1$.

Then we have

$$\begin{aligned} d_X(X_i) &= a_{X,i} X_i \qquad & 1 \le i \le l \\ d_X(X_i) &= 0 & l+1 \le i \le l+m \end{aligned}$$

and we have $d_X(\mathfrak{g}) \subset \mathfrak{n}$ and $d_X(\bar{\mathfrak{n}}) \subset \mathfrak{n}$. This implies $[\bar{\mathfrak{g}}, \bar{\mathfrak{g}}] \subset \mathfrak{n}$. In particular, $\bar{\mathfrak{n}}$ is an ideal of $\bar{\mathfrak{g}}$. Since $\bar{\mathfrak{n}}$ consists of nilpotent elements, $\bar{\mathfrak{n}}$ is a nilpotent ideal. By $\bar{\mathfrak{g}} = \{d_X + Y - d_Y | X, Y \in \mathfrak{g}\}$, we have $\bar{\mathfrak{g}} = \mathrm{Im} f \ltimes \bar{\mathfrak{n}}$. □

By this proposition, we have the inclusion $i : \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ given by $i(X) = d_X + X - d_X$ for $X \in \mathfrak{g}$.

4.2. **Constructions of algebraic hulls of simply connected solvable Lie groups.** Let G be a simply connected solvable Lie group and $\mathfrak{g}$ be the Lie algebra of G. Let N be the subgroup of G which corresponds to the nilradical $\mathfrak{n}$ of $\mathfrak{g}$. Consider the injection $i\colon \mathfrak{g} \to \mathrm{Im} f \ltimes \bar{\mathfrak{n}} \subset D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ constructed in the last subsection. Let $\bar{N}$ be the simply connected Lie group which corresponds to $\bar{\mathfrak{n}}$. Since the Lie algebra of $\mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ is $D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$, we have the Lie group homomorphism $I : G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ induced by the injective homomorphism $i : \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$.

Lemma 4.4. *The homomorphism $I : G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ is injective.*

Proof. Since the restriction of $i : \mathfrak{g} \to D(\bar{\mathfrak{n}}) \ltimes \bar{\mathfrak{n}}$ on $\mathfrak{n}$ is injective, the restriction $I : G \to \mathrm{Aut}(\bar{N}) \ltimes \bar{N}$ on N is also injective. Let T_f be the subgroup of $\mathrm{Aut}(\bar{N})$ which corresponds to $\mathrm{Im} f$. We have $I : G \to T_f \ltimes \bar{N}$. By Proposition 4.3, $\bar{\mathfrak{g}}/\mathfrak{n} = \mathrm{Im} f \oplus \bar{\mathfrak{n}}/\mathfrak{n}$. So we have $I : G/N \to T_f \times \bar{N}/N$ and it is sufficient to show this map is injective. Let $j : \mathrm{Im} f \oplus \bar{\mathfrak{n}}/\mathfrak{n} \to \bar{\mathfrak{n}}/\mathfrak{n}$ be the projection and $J : T_f \times \bar{N}/N \to \bar{N}/N$ be the homomorphism which corresponds to j. Since the composition

$$j \circ i(X \mod \mathfrak{n}) = X - d_X \mod \mathfrak{n}$$

is surjective, $j \circ i : \mathfrak{g}/\mathfrak{n} \to \bar{\mathfrak{n}}/\mathfrak{n}$ is an isomorphism. Since G/N and $\bar{N}/N$ are simply connected abelian groups, $J \circ I : G/N \to \bar{N}/N$ is also an isomorphism. Hence $I : G/N \to T_f \times \bar{N}/N$ is injective. □

We have the unipotent $\mathbb{R}$-algebraic group $\bar{\mathbf{N}}$ with $\bar{\mathbf{N}}(\mathbb{R}) = \bar{N}$. We identify $\mathrm{Aut_a}(\bar{\mathbf{N}})$ with $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$ and $\mathrm{Aut_a}(\bar{\mathbf{N}})$ has the $\mathbb{R}$-algebraic group structure with $\mathrm{Aut}_a(\bar{\mathbf{N}})(\mathbb{R}) = \mathrm{Aut}(N)$. So we have the $\mathbb{R}$-algebraic group $\mathrm{Aut_a}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}$. By the above lemma, we have the injection $I : G \to \mathrm{Aut}(N) \ltimes N = \mathrm{Aut_a}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}(\mathbb{R})$. Let $\mathbf{G}$ be the Zariski-closure of $I(G)$ in $\mathrm{Aut_a}(\bar{\mathbf{N}}) \ltimes \bar{\mathbf{N}}$.

Lemma 4.5. *We have $\mathbf{U}(\mathbf{G}) = \bar{\mathbf{N}}$.*

Proof. Let $\mathbf{T}$ be the Zariski-closure of T_f in $\mathrm{Aut}\,\bar{\mathbf{N}}$. Then $\mathbf{G} \subset \mathbf{T} \ltimes \bar{\mathbf{N}}$. Since $\mathbf{G}$ is connected solvable and $\mathbf{T}$ consists of semi-simple automorphisms, we have $\mathbf{U}(\mathbf{G}) = \mathbf{G} \cap \bar{\mathbf{N}}$. By this, it is sufficient to show $\dim \mathbf{U}(\mathbf{G}) = \dim G$. Let $\mathbf{N}$ be the Zariski-closure of $I(N)$. By $I(N) \subset \bar{N}$, we have $\mathbf{U}(\mathbf{G})/\mathbf{N} = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Thus it is sufficient to show $\dim \mathbf{U}(\mathbf{G}/\mathbf{N}) = \dim G/N$. Consider the induced map $I : G/N \to T_f \times \bar{N}/N$ as the proof of Lemma 4.4. The Zariski-closure of $I(G/N)$ in $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N}$ is $\mathbf{G}/\mathbf{N}$. Since $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N}$ is commutative, the projection $\mathbf{T} \times \bar{\mathbf{N}}/\mathbf{N} \to \bar{\mathbf{N}}/\mathbf{N}$ is an $\mathbb{R}$-algebraic group homomorphism,

and hence the Zariski-closure of $J \circ I(G/N)$ in $\bar{\mathbf{N}}/\mathbf{N}$ is $\mathbf{U}(\mathbf{G}/\mathbf{N})$. Otherwise in the proof of Lemma 4.4 we showed that $J \circ I : G/N \to \bar{N}/N$ is isomorphism. This implies $\bar{\mathbf{N}}/\mathbf{N} = \mathbf{U}(\mathbf{G}/\mathbf{N})$. Hence the lemma follows. □

By this lemma we have the following proposition.

Proposition 4.6. $\mathbf{G}$ *is the algebraic hull of* G *and the Lie algebra of the unipotent hull* $\mathbf{U}_G$ *is* $\bar{\mathfrak{n}}_{\mathbb{C}}$.

Proof. We show that $\mathbf{G}$ satisfies the properties of the algebraic hull of G. We have $\dim \mathbf{U}(\mathbf{G}) = \dim \bar{\mathbf{N}} = \dim G$. Let $(t,\ x) \in Z_{\mathbf{G}}(\mathbf{U}(\mathbf{G})) \subset \mathrm{Aut_a}\bar{\mathbf{N}} \ltimes \bar{\mathbf{N}}$. Since $\mathbf{U}(\mathbf{G}) = \mathbf{N}$ and t is a semi-simple automorphism, we have $t(y) = y$ for any $y \in \bar{\mathbf{N}}$. So we have $t = \mathrm{id}_{\bar{\mathbf{N}}}$. We have $Z_{\mathbf{G}}(\mathbf{U}(\mathbf{G})) \subset \mathbf{U}(\mathbf{G})$. Hence the proposition follows.

□

4.3. **Proof of Theorem 1.2.** To prove the theorem, we first show the following lemma.

Lemma 4.7. *Let* $\mathfrak{g}$ *be a solvable Lie algebra and* $\mathfrak{n}$ *the nilradical of* $\mathfrak{g}$. *If* $\mathfrak{n}$ *is abelian and for the extension*

$$0 \to \mathfrak{n} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{n} \to 0$$

the action of $\mathfrak{g}/\mathfrak{n}$ *on* $\mathfrak{n}$ *is semi-simple, then we have a semi-direct decomposition* $\mathfrak{g} = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that* ϕ *is semi-simple.*

Proof. Let V_0 be the weight vector space of the action of $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{n}$ with weight 0. Since $\mathfrak{g}/\mathfrak{n}$ acts semi-simply, we have a direct sum $\mathfrak{n} = \mathfrak{n}' \oplus V_0$ of $\mathfrak{g}/\mathfrak{n}$-modules. Then we have $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{n}'] = \mathfrak{n}'$ and $[\mathfrak{g}, V_0] = 0$. This implies that V_0 is an ideal of $\mathfrak{g}$. Choose a subvector space $\mathfrak{g}' \subset \mathfrak{g}$ so that we have $\mathfrak{g} = \mathfrak{g}' \oplus V_0$ as a direct sum of vector spaces and $\mathfrak{n}' \subset \mathfrak{g}'$. Since we have

$$\mathfrak{n}' = [\mathfrak{g}, \mathfrak{n}'] \subset [\mathfrak{g}, \mathfrak{g}'] \subset [\mathfrak{g}, \mathfrak{g}] = \mathfrak{n}',$$

$\mathfrak{g}'$ is an ideal of $\mathfrak{g}$, $\mathfrak{n}'$ is an ideal of $\mathfrak{g}'$ and $\mathfrak{g}' \oplus V_0$ is also a direct sum of Lie algebras. Hence it is sufficient to show the extension

$$0 \to \mathfrak{n}' \to \mathfrak{g}' \to \mathfrak{g}'/\mathfrak{n}' \to 0$$

sprits. By the construction of $\mathfrak{g}'$ and $\mathfrak{n}'$, $\mathfrak{n}'$ does not contain a trivial $\mathfrak{g}'/\mathfrak{n}'$-module. By the result in [6], we have $H^2(\mathfrak{g}'/\mathfrak{n}', \mathfrak{n}') = \{0\}$. Hence the lemma follows.

□

Theorem 4.8. *Let* G *be a simply connected solvable Lie group. Then* $\mathbf{U}_G$ *is abelian if and only if* $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ *such that the action* $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ *is semi-simple.*

Proof. Consider the inclusion $i : \mathfrak{g} \to \mathrm{Im} f \ltimes \bar{\mathfrak{n}}$. By the above argument, the Lie algebra of $\mathbf{U}_G$ is $\bar{\mathfrak{n}}_{\mathbb{C}}$. Suppose $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \mathrm{Aut}\,\mathbb{R}^m$ is semi-simple. It is sufficient to show $\bar{\mathfrak{n}} = \{X - d_X | X \in \mathfrak{g}\} \subset \mathrm{Im} f \ltimes \bar{\mathfrak{n}}$ is an abelian Lie algebra. Let $X, Y \in \mathfrak{g}$ and $X = X_1 + X_2$, $Y = Y_1 + Y_2$ be the decompositions induced by the semi-direct product $\mathfrak{g} = \mathbb{R}^n \ltimes_{\phi_*} \mathbb{R}^m$. Then we have $d_{X_2} = 0$, $d_{Y_2} = 0$, $[X_1, Y_1] = 0$ and $[X_2, Y_2] = 0$ by the assumption. Hence we have

$$[X - d_X, Y - d_Y] = [X_1, Y_2] + [X_2, Y_1] - d_{X_1}(Y_2) + d_{Y_1}(X_2).$$

Since the action ϕ_* is semi-simple, we have $d_{X_1}(Y_2) = [X_1, Y_2]$ and $d_{Y_1}(X_2) = [Y_1, X_2]$. Therefore we have $[X - d_X, Y - d_Y] = 0$. This implies $\bar{\mathfrak{n}}$ is abelian.

Conversely we assume $\mathbf{U}_G$ is abelian. By the assumption, $\bar{\mathfrak{n}}$ is abelian. Since $i(\mathfrak{n}) \subset \bar{\mathfrak{n}}$, $\mathfrak{n}$ is abelian. By Lemma 4.7 it is sufficient to show that the action $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{n}$ is semi-simple. Suppose ad_X on $\mathfrak{n}$ is not semi-simple. Then $\mathrm{ad}_X - d_X$ on $\mathfrak{n}$ is not trivial. Since we have $\bar{\mathfrak{n}} = \{X - d_X | X \in \mathfrak{g}\} \subset \mathrm{Im} f \ltimes \bar{\mathfrak{n}}$, we have $[\bar{\mathfrak{n}}, \mathfrak{n}] \neq \{0\}$. This contradicts $\bar{\mathfrak{n}}$ is abelian. Hence we have the action $\mathfrak{g}/\mathfrak{n}$ on $\mathfrak{n}$ is semi-simple.

□

By this theorem we show the following theorem.

Theorem 4.9. *Let Γ be a torsion-free virtually polycyclic group. Then the following two conditions are equivalent:*
(1) $\mathbf{U}_\Gamma$ *is abelian.*
(2) Γ *is a finite extension group of a lattice of Lie group G such that $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ with the action $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple.*

Proof. By Theorem 3.2, we have a finite index subgroup of Γ which isomorphic to a lattice of some simply connected solvable Lie group G. By Lemma 3.5 and 3.10, we have $\mathbf{U}_\Gamma = \mathbf{U}_G$. Hence by Theorem 4.8 we have the theorem. □

5. **Abelian unipotent hulls and formality**

5.1. **Abelian unipotent hulls and formality.**

We review the definition of formality and prove Proposition 1.1.

Definition 5.1. A differential graded algebra (called DGA) is a graded $\mathbb{R}$- algebra A^* with the following properties:
(1) A^* is graded commutative, i.e.

$$y \wedge x = (-1)^{p \cdot q} x \wedge y \quad x \in A^p \quad y \in A^q.$$

(2) There is a boundary operator $d : A \to A$ of degree one such that $d \circ d = 0$ and

$$d(x \wedge y) = dx \wedge y + (-1)^p x \wedge dy \quad x \in A^p \quad y \in A^q.$$

Let A and B be DGAs. If a morphism of graded algebra $\varphi : A \to B$ satisfies $d \circ \varphi = \varphi \circ d$, we call φ a morphism of DGAs. If a morphism of DGAs induces the cohomology isomorphism, we call it a quasi-isomorphism.

Definition 5.2. A and B are weakly equivalent if there is a finite diagram of DGAs

$$A \to C_1 \leftarrow C_2 \cdots \leftarrow B$$

such that all the morphisms are quasi-isomorphisms.

Let M be a smooth manifold. The De Rham complex $A^*(M)$ of M is the basic example of a DGA. The cohomology algebra $H^*(M, \mathbb{R})$ is a DGA with $d = 0$.

Definition 5.3. A smooth manifold M is formal if $A^*(M)$ and $H^*(M, \mathbb{R})$ are weakly equivalent.

Proposition 5.4. *Let Γ be a torsion-free virtually polycyclic group. If the unipotent hull U_Γ is abelian, the standard Γ-manifold M_Γ is formal.*

Proof. We denote U, T and $(\bigwedge \mathfrak{u}^*)^T$ as Section 3.4. If the k-unipotent hull of Γ is abelian, $(\bigwedge \mathfrak{u}^*, d) = (\bigwedge \mathfrak{u}^*, 0)$. By Theorem 3.12, we have the diagram of DGAs

$$A^*(M_\Gamma) \leftarrow ((\bigwedge \mathfrak{u}^*)^T) = H^*(M_\Gamma)$$

such that the map $A^*(M_\Gamma) \leftarrow ((\bigwedge \mathfrak{u}^*)^T)$ is a quasi-isomorphism. Hence the proposition follows. $\square$

By the last section we have the following corollary.

Corollary 5.5. *If Γ satisfies the condition (2) in Theorem 4.9, then $K(\Gamma, 1)$ is formal.*

5.2. **Relations to the geometries of 3-dimensional manifolds.** We give examples of formal spaces which relate to 3-dimensional geometry. See [17] for the general theory of 3-dimensional geometries.

Corollary 5.6. *Let M be a compact 3-dimensional manifold. If the geometric structure of M is E^3 or Sol, then M is formal.*

Proof. For E^3, for any lattice Γ in $\mathrm{Isom}(E^3) \cong \mathbb{R}^3 \rtimes O(3)$, the intersection $\Gamma \cap \mathbb{R}^3$ is a lattice of $\mathbb{R}^3$ and a finite index subgroup of Γ by Bieberbach's first theorem. This implies M is formal if the geometric structure of M is E^3. For Sol, Sol is the Lie group $G = \mathbb{R}_\phi \ltimes \mathbb{R}^2$ such that $\phi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix}$ with an invariant metric. Let Γ be a discrete subgroup of $\mathrm{Isom}(Sol)$ such that $\Gamma\backslash G$ is compact. Since the identity component of $\mathrm{Isom}(Sol)$ is G and it is a finite index normal subgroup of $\mathrm{Isom}(Sol)$(see [17]), $\Gamma \cap G$ is a finite index subgroup of Γ and $\Gamma \cap G\backslash G$ is compact. Hence Γ is a finite extension of a lattice of G. By Corollary 5.5, we have the corollary. $\square$

6. Relations to Kähler Structures

6.1. The hard Lefschetz property.

We review the definition of the hard Lefschetz property and prove Proposition 1.5.

Definition 6.1. Let (M,ω) be a $2n$-dimensional symplectic manifold. We say that (M,ω) satisfies the hard Lefschetz property if the linear map

$$[\omega^{n-i}]\wedge : H^i(M,\mathbb{R}) \to H^{2n-i}(M,\mathbb{R})$$

is an isomorphism for any $0 \le i \le n$.

Theorem 6.2. ([14]) *Compact Kähler manifolds satisfy the hard Lefschetz properties.*

Proposition 6.3. *Let Γ be a torsion-free virtually polycyclic group. Suppose the standard Γ-manifold M_Γ admits a symplectic structure. If the unipotent hull $\mathbf{U}_\Gamma$ is abelian, M_Γ satisfies the hard Lefschetz property.*

Proof. As in Section 3.4, we have the sub-DGA $(\bigwedge \mathfrak{u}^*)^T$ with $d = 0$ in $A^*(M_\Gamma)$ and the isomorphism $(\bigwedge \mathfrak{u}^*)^T \cong H^*(M_\Gamma,\mathbb{R})$. For a symplectic form of ω on M_Γ, we have $\omega_0 \in (\bigwedge \mathfrak{u}^*)^T$ which is cohomologous to ω. Since $\omega^n \neq 0$ for $2n = \dim \mathfrak{u} = \dim M_\Gamma$, ω_0 is a symplectic form on the vector space $\mathfrak{u}$. Since

$$\omega_0^{n-i}\wedge : \bigwedge \mathfrak{u}^i \to \bigwedge \mathfrak{u}^{2n-i}$$

is injective for any $0 \le i \le n$ by the hard Lefschetz property of a torus,

$$\omega_0^{n-i}\wedge : (\bigwedge \mathfrak{u}^i)^T \to (\bigwedge \mathfrak{u}^{2n-i})^T$$

is also injective and so

$$[\omega^{n-i}]\wedge : H^i(M_\Gamma,\mathbb{R}) \to H^{2n-i}(M_\Gamma,\mathbb{R})$$

is injective and thus it is an isomorphism by the Poincaré duality. Hence we have the proposition. $\square$

Corollary 6.4. *Suppose M_Γ admits a symplectic structure. If Γ satisfies the condition (2) in Theorem 1.2, M_Γ satisfies the hard Lefschetz property.*

6.2. **Formal solvmanifolds satisfying the hard Lefschetz property but not admitting Kähler structure.** In [1], Arapura showed that the fundamental groups of compact Kähler solvmanifolds are virtually abelian. Let G be a simply connected solvable Lie group. We call G type (I) if for any $g \in G$ the all eigenvalues of the adjoint operator Ad_g have absolute value 1. In [2] it was proved that a lattice of a simply connected solvable Lie group G is virtually nilpotent if and only if G is type (I). In [3] Baues proved every compact solvmanifold with the fundamental group Γ is diffeomorphic to the standard Γ-manifold. Hence we have the following corollary.

Corollary 6.5. *Let $G = \mathbb{R}^n \ltimes_\phi \mathbb{R}^m$ such that $\phi : \mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^m)$ is semi-simple and G is not type (I). Then for any lattice Γ of G, G/Γ is a formal solvmanifold which does not admit a Kähler structure. If G/Γ admits a symplectic structure, G/Γ satisfies the hard Lefschetz property.*

6.3. **Examples.** Earlier, in [8] Fernandez, and Gray constructed examples of formal solvmanifolds satisfying the hard Lefschetz property not admitting a Kähler structure. For a Lie group $G = \mathbb{R} \ltimes_\phi \mathbb{R}^2$ with $\phi(t) = \begin{pmatrix} e^{kt} & 0 \\ 0 & e^{-kt} \end{pmatrix}$, they showed that for a lattice Γ of G the manifold $G/\Gamma \times S^1$ is such an example. By the result of this paper we generalize this construction.

Example.1(Generalizations of Fernandez and Gray's examples)
Let $G = \mathbb{R} \ltimes_\phi \mathbb{R}^{2(m+n)+1}$ such that

$$\phi(t) = \begin{pmatrix} e^{a_1 t} & 0 \\ 0 & e^{-a_1 t} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} e^{a_m t} & 0 \\ 0 & e^{-a_m t} \end{pmatrix}$$
$$\oplus \begin{pmatrix} \cos b_1 t & -\sin b_1 t \\ \sin b_1 t & \cos b_1 t \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} \cos b_n t & -\sin b_n t \\ \sin b_n t & \cos b_n t \end{pmatrix} \oplus (1)$$

for $a_i, b_i \in \mathbb{R}$.
Then the cochain complex $(\bigwedge \mathfrak{g}^*, d)$ of the Lie algebra of G is given by:

$$\mathfrak{g}^* = \langle \tau, x_i, y_i, z_j, w_j, \sigma \rangle,$$

$$d\tau = d\sigma = 0,$$

$$dx_i = -a_i\tau \wedge x_i,\ dy_i = a_i\tau \wedge y_i, \quad (1 \le i \le m),$$

$$dz_j = b_j\tau \wedge w_j,\ dw_j = -b_j\tau \wedge z_j, \quad (1 \le j \le n).$$

We have an invariant symplectic form $\omega = \tau\wedge\sigma + \sum_i^m x_i\wedge y_i + \sum_j^n z_j\wedge w_j$ Hence for any lattice Γ, G/Γ is formal and satisfies the hard Lefschetz property. If some a_i is not zero, G/Γ does not admit a Kähler metric.

Example.2(complex example)

Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ with $\phi(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix}$. Then the cochain complex $(\bigwedge \mathfrak{g}^*, d)$ of the Lie algebra of G is given by:

$$\mathfrak{g}^* = \langle x_1, x_2, y_1, y_2, z_1, z_2 \rangle,$$

$$dx_1 = dx_2 = 0,$$

$$dy_1 = -x_1 \wedge y_1 + x_2 \wedge y_2,\ dy_2 = -x_2 \wedge y_1 - x_1 \wedge y_2,$$

$$dz_1 = x_1 \wedge z_1 - x_2 \wedge z_2,\ dz_2 = x_1 \wedge z_2 + x_2 \wedge z_1.$$

We have an invariant symplectic form $\omega = x_1 \wedge x_2 + z_1 \wedge y_1 + y_2 \wedge z_2$. In [10], it was shown that G has some lattices. For any lattice Γ, G/Γ is complex, symplectic with the hard Lefschetz property and formal but not Kähler.

7. **Remarks**

In this Section we give an example of a formal standard Γ-manifold with the hard Lefschetz property such that U_Γ is not abelian. In addition this is also an example of formal manifold satisfying the hard Lefschetz property such that it is finitely covered by a non-formal manifold not satisfying the hard Lefschetz property. We notice that compact manifolds finitely covered by non-Kähler manifolds are not Kähler.

Let $\Gamma = \mathbb{Z} \ltimes_\phi \mathbb{Z}^2$ such that for $t \in \mathbb{Z}$

$$\phi(t) = \begin{pmatrix} (-1)^t & (-1)^t t \\ 0 & (-1)^t \end{pmatrix}.$$

Lemma 7.1. *The algebraic hull of* Γ *is given by* $\mathbf{H_\Gamma} = \{\pm \mathbf{1}\} \ltimes \mathbf{U_3}(\mathbb{C})$ *such that*

$$(-1) \cdot \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & (-1)z \\ 0 & 1 & (-1)y \\ 0 & 0 & 1 \end{pmatrix}$$

Proof. We have the inclusion

$$\Gamma \cong \left((-1)^x, \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right) \subset \{\pm 1\} \ltimes \mathbf{U_\Gamma}.$$

Then Γ is Zariski-dense in $\{\pm 1\} \ltimes \mathbf{U_\Gamma}$ and $\operatorname{rank}\Gamma = 3 = \dim \mathbf{U}_\Gamma$. Since the action of $\{\pm 1\}$ on $\mathbf{U}_\Gamma$ is faithful, we have $Z_{\mathbf{H}_\Gamma}(\mathbf{U}_\Gamma) \subset \mathbf{U}_\Gamma$. Hence the lemma follows. □

We have $\mathbf{H}_\Gamma(\mathbb{R}) = \{\pm 1\} \ltimes U_\Gamma$ such that $U_\Gamma = \mathbf{U}_3(\mathbb{R})$. Let $\mathfrak{u}$ be the Lie algebra of U_Γ. We have $\mathfrak{u} = \langle X_1, X_2, X_3 \rangle$ such that the bracket is given by

$$[X_1, X_2] = -[X_2, X_1] = X_3.$$

The $\{\pm 1\}$-action on $\mathfrak{u}$ is given by

$$(-1) \cdot X_1 = X_1,\ (-1) \cdot X_i = -X_i \qquad i = 2, 3$$

Let x_1, x_2, x_3 be the basis of $\mathfrak{u}^*$ which is dual to X_1, X_2, X_3. Then the DGA $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}$ is the subalgebra of $\bigwedge \mathfrak{u}^*$ generated by $\{x_1, x_2 \wedge x_3\}$ and the derivation on $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}$ is trivial. Let M_Γ be the standard Γ-manifold. Then by Theorem 3.12, we have the quasi-isomorphism $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} \to A^*(M_\Gamma)$. Since the derivation on $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}}$ is trivial, we have the isomorphism $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} \cong H^*(M)$. Hence we have:

Proposition 7.2. *M_Γ is formal.*

Remark 1. Since U_Γ is not abelian, the converse of Proposition 5.4 is not true.

Remark 2. We have the finite index subgroup $2\mathbb{Z} \ltimes \mathbb{Z}^2$ which is nilpotent. So Γ is virtually nilpotent but not virtually abelian. By the result of [9], $K(2\mathbb{Z} \ltimes \mathbb{Z}^2, 1)$ is not formal. But for the finite extension group Γ, $K(\Gamma, 1)$ is formal.

Remark 3. Since $\{\pm 1\}$ acts isometrically on U_Γ with the invariant metric, M_Γ has Nil structure. So we have a formal 3-dimensional compact manifold which has Nil structure.

Let $\Delta = \Gamma \times \mathbb{Z}$. Then we have $H_\Delta = H_\Gamma \times \mathbb{R}$ and $U_\Delta = U_\Gamma \times \mathbb{R}$. As above we have the quasi-isomorphism inclusion $(\bigwedge \mathfrak{u}^*)^{\{\pm 1\}} \otimes \bigwedge(y) \subset A^*(M_\Delta)$. Let $\omega = x_1 \wedge y + x_2 \wedge x_3$. Then ω is a symplectic form on M_Δ. Since $H^1(M_\Delta, \mathbb{R}) \cong \langle x_1, y \rangle$ and $H^3(M_\Delta, \mathbb{R}) \cong \langle x_1 \wedge x_2 \wedge x_3, x_2 \wedge x_3 \wedge y \rangle$, the linear map $[\omega]\wedge : H^1(M_\Delta, \mathbb{R}) \to H^3(M_\Delta, \mathbb{R})$ is an isomorphism and hence we have the following proposition.

Proposition 7.3. *$M_\Gamma \times S^1$ satisfies the hard Lefschetz property.*

Remark 4. Δ is a finite extension group of the non-abelian nilpotent group $2\mathbb{Z}\ltimes\mathbb{Z}^2\times\mathbb{Z}$ as remark 2. By the result of [4], a compact $K(2\mathbb{Z}\ltimes\mathbb{Z}^2\times\mathbb{Z},1)$-manifold is not a Lefschetz 4-manifold. Thus M_Δ is a example of a Lefschetz 4-manifold with non-Lefschetz finite covering space. In [11, Example 3.4], Lin showed the existence of Lefschetz 4-manifolds with non-Lefschetz finite covering space. M_Δ is a simpler and more constructive example.

Acknowledgements.
The author would like to express his gratitude to Toshitake Kohno for helpful suggestions and stimulating discussions. He would also like to thank Katsuhiko Kuribayashi and Keizo Hasegawa for their active interests in this paper. This research is supported by JSPS Research Fellowships for Young Scientists.

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